Tuesday, October 27, 2015

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Problem 79

Problem. Use integration by parts to verify the reduction formula

$$\int \sin^n x\, dx = -\frac{\sin^{n-1} x \cos x}{n} + \frac{n-1}{n} \int \sin^{n-2} x\, dx.$$

Solution. Let $u = \sin^{n-1} x$ and $dv = \sin x\, dx$. Then $du = (n-1)\sin^{n-2} x \cos x\, dx$ and $v = -\cos x$.

$$\begin{aligned}
\int \sin^n x\, dx &= (\sin^{n-1} x)(-\cos x) + (n-1) \int \cos x \cdot \sin^{n-2} x \cos x\, dx \\
&= -\sin^{n-1} x \cos x + (n-1) \int \sin^{n-2} x \cos^2 x\, dx \\
&= -\sin^{n-1} x \cos x + (n-1) \int \sin^{n-2} x (1 - \sin^2 x)\, dx \\
&= -\sin^{n-1} x \cos x + (n-1) \int \sin^{n-2} x\, dx - (n-1) \int \sin^n x\, dx.
\end{aligned}$$

Add $(n-1)\int \sin^n x\, dx$ to both sides and divide by n.

$$\begin{aligned}
n \int \sin^n x\, dx &= -\sin^{n-1} x \cos x + (n-1) \int \sin^{n-2} x\, dx \\
&= -\frac{\sin^{n-1} x \cos x}{n} + \frac{n-1}{n} \int \sin^{n-2} x\, dx.
\end{aligned}$$

Problem 80

Problem. Use integration by parts to verify the reduction formula

$$\int \cos^n x\, dx = \frac{\cos^{n-1} x \sin x}{n} + \frac{n-1}{n} \int \cos^{n-2} x\, dx.$$

Solution. Let $u = \cos^{n-1} x$ and $dv = \cos x\, dx$. Then $du = -(n-1)\cos^{n-2} x \sin x\, dx$

and $v = \sin x$.

$$\begin{aligned}
\int \cos^n x \, dx &= \cos^{n-1} x \sin x + (n-1) \int \sin x \cdot \cos^{n-2} x \sin x \, dx \\
&= \cos^{n-1} x \sin x + (n-1) \int \cos^{n-2} x \sin^2 x \, dx \\
&= \cos^{n-1} x \sin x + (n-1) \int \cos^{n-2} x (1 - \cos^2 x) \, dx \\
&= \cos^{n-1} x \sin x + (n-1) \int \cos^{n-2} x \, dx - (n-1) \int \cos^n x \, dx.
\end{aligned}$$

Add $(n-1) \int \cos^n x \, dx$ to both sides and divide by n.

$$\begin{aligned}
n \int \cos^n x \, dx &= \cos^{n-1} x \sin x + (n-1) \int \cos^{n-2} x \, dx \\
&= \frac{\cos^{n-1} x \sin x}{n} + \frac{n-1}{n} \int \cos^{n-2} x \, dx.
\end{aligned}$$

Problem 81

Problem. Use integration by parts to verify the reduction formula

$$\int \cos^m x \sin^n x \, dx = -\frac{\cos^{m+1} x \sin^{n-1} x}{m+n} + \frac{n-1}{m+n} \int \cos^m x \sin^{n-2} x \, dx.$$

Solution. Let

$$\begin{aligned}
u &= \cos^m x \sin^{n-1} x \\
dv &= \sin x \, dx.
\end{aligned}$$

Then

$$\begin{aligned}
du &= ((m \cos^{m-1} x (-\sin x))(\sin^{n-1} x) + (\cos^m x)((n-1) \sin^{n-2} x \cos x) \, dx \\
&= (-m \cos^{m-1} x \sin^n x + (n-1) \cos^{m+1} x \sin^{n-2}) \, dx, \\
v &= -\cos x.
\end{aligned}$$

Then

$$\begin{aligned}
\int \cos^m x \sin^n x \, dx = (\cos^m x \sin^{n-1} x)(-\cos x) - \int (-\cos x)((m \cos^{m-1} x (-\sin x))(\sin^{n-1} x) \\
+ (\cos^m x)((n-1) \sin^{n-2} x \cos x) \, dx \\
= -\cos^{m+1} x \sin^{n-1} x - m \int \cos^m x \sin^n x \, dx + (n-1) \int \cos^{m+2} x \sin^{n-2} x
\end{aligned}$$

$$(m+1) \int \cos^m x \sin^n x \, dx = -\cos^{m+1} x \sin^{n-1} x + (n-1) \int \cos^{m+2} x \sin^{n-2} x \, dx.$$

In the last integral, rewrite $\cos^{m+2} x$ as $\cos^m x(1-\sin^2 x)$ and continue.

$$\begin{aligned}(m+1)\int \cos^m x \sin^n x\,dx &= -\cos^{m+1} x \sin^{n-1} x + (n-1)\int \cos^m x(1-\sin^2 x)\sin^{n-2} x\,dx \\ &= -\cos^{m+1} x \sin^{n-1} x + (n-1)\int \cos^m x \sin^{n-2} x\,dx - (n-1)\int \cos^m x \sin \\ (m+n)\int \cos^m x \sin^n x\,dx &= -\cos^{m+1} x \sin^{n-1} x + (n-1)\int \cos^m x \sin^{n-2} x\,dx, \\ \int \cos^m x \sin^n x\,dx &= -\frac{\cos^{m+1} x \sin^{n-1} x}{m+n} + \frac{n-1}{m+n}\int \cos^m x \sin^{n-2} x\,dx.\end{aligned}$$

Problem 83

Problem. Use the results of Exercises 79-82 to find the integral $\int \sin^5 x\,dx$.

Solution.

$$\begin{aligned}\int \sin^5 x\,dx &= -\frac{\sin^3 x \cos x}{5} + \frac{4}{5}\int \sin^3 x\,dx \\ &= -\frac{\sin^3 x \cos x}{5} + \frac{4}{5}\left(-\frac{\sin x \cos x}{3} + \frac{2}{3}\int \sin x\,dx\right) \\ &= -\frac{\sin^3 x \cos x}{5} + \frac{4}{5}\left(-\frac{\sin x \cos x}{3} + \frac{2}{3}(-\cos x)\right) \\ &= -\frac{\sin^3 x \cos x}{5} - \frac{4\sin x \cos x}{15} - \frac{8\cos x}{15} + C.\end{aligned}$$

Problem 84

Problem. Use the results of Exercises 79-82 to find the integral $\int \cos^4 x\,dx$.

Solution.

$$\begin{aligned}\int \cos^4 x\,dx &= \frac{\cos^3 x \sin x}{4} + \frac{3}{4}\int \cos^2 x\,dx \\ &= \frac{\cos^3 x \sin x}{4} + \frac{3}{4}\left(\frac{\cos x \sin x}{2} + \frac{1}{2}\int\,dx\right) \\ &= \frac{\cos^3 x \sin x}{4} + \frac{3}{4}\left(\frac{\cos x \sin x}{2} + \frac{1}{2}x\right) \\ &= \frac{\cos^3 x \sin x}{4} + \frac{3\cos x \sin x}{8} + \frac{3}{8}x + C.\end{aligned}$$

Problem 85

Problem. Use the results of Exercises 79-82 to find the integral $\int \sec^4(2\pi x/5)\, dx$.

Solution. To simplify the algebra, begin with the substitution $u = \frac{2\pi x}{5}$, $du = \frac{2\pi}{5}\, dx$. Then we have

$$\begin{aligned}\int \sec^4(2\pi x/5)\, dx &= \frac{5}{2\pi}\int \sec^4 u\, du \\ &= \frac{5}{2\pi}\left(\frac{1}{3}\sec^2 u \tan u + \frac{2}{3}\int \sec^2 u\, du\right) \\ &= \frac{5}{2\pi}\left(\frac{1}{3}\sec^2 u \tan u + \frac{2}{3}\tan u\right) \\ &= \frac{5}{2\pi}\left(\frac{1}{3}\sec^2 \frac{2\pi x}{5} \tan \frac{2\pi x}{5} + \frac{2}{3}\tan \frac{2\pi x}{5}\right) \\ &= \frac{5}{6\pi}\sec^2 \frac{2\pi x}{5} \tan \frac{2\pi x}{5} + \frac{5}{3\pi}\tan \frac{2\pi x}{5} + C.\end{aligned}$$

Problem 86

Problem. Use the results of Exercises 79-81 to find the integral $\int \sin^4 x \cos^2 x\, dx$.

Solution.

$$\int \sin^4 x \cos^2 x\, dx = -\frac{\cos^5 x \sin x}{6} + \frac{1}{6}\int \cos^4 x\, dx.$$

Now use the result of Exercise 84 to finish the problem.

$$\begin{aligned}\int \sin^4 x \cos^2 x\, dx &= -\frac{\cos^5 x \sin x}{6} + \frac{1}{6}\left(\frac{\cos^3 x \sin x}{4} + \frac{3\cos x \sin x}{8} + \frac{3}{8}x\right) \\ &= .-\frac{\cos^5 x \sin x}{6} + \frac{\cos^3 x \sin x}{24} + \frac{\cos x \sin x}{16} + \frac{1}{16}x + C.\end{aligned}$$